MATH 504 HOMEWORK 7

Due Friday, December 7.

Problem 1. *Let $\mathbb{P}$ be a poset such that for every $p \in \mathbb{P}$, there are incompatible $q, r \leq p$. Suppose G is $\mathbb{P}$-generic. Show that $G \times G$ is not $\mathbb{P} \times \mathbb{P}$-generic.*

Problem 2. *Let T be a normal Suslin tree, and let $(\mathbb{P}_T, <) = (T, >)$. Show that although $\mathbb{P}_T$ has the countable chain condition, $\mathbb{P}_T \times \mathbb{P}_T$ does not. Hint: for every $x \in T$, pick two immediate successors p_x, q_x of x. Look at the set $\{(p_x, q_x) \mid x \in T\} \subset \mathbb{P}_T \times \mathbb{P}_T$.*

Problem 3. *Let κ be a regular uncountable cardinal and $\mathbb{P}$ be a κ-closed poset. Show that $\mathbb{P}$ preserves stationary subsets of κ, i.e. if $S \subset \kappa$ is stationary in the ground model, then S remains stationary in any $\mathbb{P}$-generic extension.*
Hint: Given S, a name $\dot{C}$, and p, such that $p \Vdash$ "$\dot{C}$ is a club subset of κ", show there is a sequence in the ground model $\langle p_\alpha, \gamma_\alpha \mid \alpha < \kappa\rangle$, such that:

- *each $p_\alpha \leq p$ and $p_\alpha \Vdash \gamma_\alpha \in \dot{C}$.*
- *$\langle \gamma_\alpha \mid \alpha < \kappa\rangle$ is a club in κ,*

We say that a poset is **κ-distributive** if whenever $p \Vdash \dot{f} : \kappa \to \mathrm{ON}$, then there is some $q \leq p$ and a function g in the ground model, such that $q \Vdash \dot{f} = \check{g}$.

Problem 4. (1) *Show that if $\mathbb{P}$ is κ^+-closed, then it is κ-distributive.*
(2) *Show that being κ-distributive is equivalent to the following property: If $\langle D_\alpha \mid \alpha < \kappa\rangle$ is a family of κ-many dense open sets, then $\bigcap_{\alpha<\kappa} D_\alpha$ is also open dense.*
(3) *Suppose that $\mathbb{P}$ is κ-distributive. Show that $\mathbb{P}$ preserves κ^+ (it also preserves κ).*

Problem 5. *Let $S \subset \omega_1$ be a stationary set. Define $\mathbb{P} := \{p \subset S \mid p \text{ is closed and bounded}\}$, and set $p \leq q$ if p end extends q i.e. for some α, $p \cap \alpha = q$.*
(1) *Show that $\mathbb{P}$ is countably distributive, i.e. if $p \Vdash \dot{f} : \omega \to ON$, then there is some $q \leq p$ and a function g in the ground model, such that $q \Vdash \dot{f} = \check{g}$. Note that this implies that $\mathbb{P}$ adds no countable subsets of ω_1, and hence it preserves ω_1.*
(2) *Suppose that $T := S \setminus \omega_1$ is also stationary. Let G be a $\mathbb{P}$-generic filter. Show that in $V[G]$, T is nonstationary.*

Remark 1. The above is an example of a forcing that destroys a stationary set, without collapsing cardinals. On the other hand you cannot use forcing to destroy a club set. More precisely, if $V \subset W$ are two models of set theory and $V \models$ “D is club in κ”, then $W \models$ “D is club in κ”. Note that in the above problem it was important that S was stationary; i.e. you cannot add a new club through a nonstationary set.